Boundary conditions for hyperbolic formulations of the Einstein equations

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Abstract. In regards to the initial-boundary value problem of the Einstein equations, we argue that the projection of the Einstein equations along the normal to the boundary yields necessary and appropriate boundary conditions for a wide class of equivalent formulations. We explicitly show that this is so for the Einstein-Christoffel formulation of the Einstein equations in the case of spherical symmetry.

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1. Introduction

Achieving long time simulations of binary black hole spacetimes remains one of the most pressing issues in numerical relativity. Numerical simulations are plagued with instabilities, the origin of which has not been isolated among a number of possible factors of relevance to stability (see for instance [1, 2, 3] and references therein). A partial list of suspected sources of instabilities includes ill-posed evolution, ill-posed constraint propagation and poor choices of boundary conditions, as well as poor choices of binary black hole data. The problem of well posedness of the evolution equations and constraint propagation in the analytic (as opposed to numerical) sense has been widely studied (see the Living Review article by Reula [4]), with the result that there is a great deal of choice of strongly hyperbolic formulations of the Einstein equations available for analytic studies (aside from numerical implementation). On the other hand, appropriate boundary values for evolution remain to be identified. Since a great deal more about the boundary value problem of a strongly hyperbolic set of equations is known than for any other kind, currently analytic investigations of appropriate boundary conditions tend to use some kind of strongly hyperbolic formulation of the Einstein equations. Relevant considerations usually run along the following lines.

The boundary values of a strongly hyperbolic system of partial differential equations split always into two sets [5]: those that are determined by the initial conditions of the problem and cannot be chosen freely, and those that are entirely arbitrary because they are determined by values of the fields outside of the region of interest. In practice, the free boundary data must be specified in order for a unique solution of the equations to be determined in the region of interest.

Suppose we have a boundary delimiting a region where we seek a solution to the Einstein equations in some strongly hyperbolic formulation. In this case one calculates data on the initial slice satisfying the constraints, and then integrates up the hyperbolic equations step by step. But, by the previous paragraph, the initial data alone are not sufficient to find a unique solution in the future of a bounded region in space. Boundary values must be prescribed as well. Suppose that we know exactly which variables' boundary values are to be considered independent of the initial data. How are we to prescribe them? Since in some sense the solution that we are seeking must satisfy all the Einstein equations, and by construction the evolution equations are being imposed, then it seems that the appropriate question to ask is: What are the boundary data that preserve the constraints?

This question is addressed by Stewart in [6] within a particular strongly hyperbolic formulation of the Einstein equations [7, 8]. The constraints themselves, considered as real functions of spacetime, propagate according to a strongly hyperbolic system of their own. This implies that vanishing values of the constraints at the initial time will propagate towards the future along characteristics. Some of the constraints propagate towards the boundary and cross out of the region of interest, whereas others propagate into the region of interest by crossing the boundary from outside. Clearly the values of the incoming constraints at the boundary are arbitrary, and one wants to have them vanish. But the vanishing of the constraints cannot be imposed along the boundaries in practice. The constraints involve derivatives of the fields across the boundaries, not just the values of the fields themselves. Stewart argues that the vanishing of the incoming constraints can in fact be used as a boundary condition on the fields when the equations are linearized around flat space. If the fields are expressed in integral form in terms of Fourier-Laplace transforms, the linearity of the differential

equations implies algebraic equations for the transforms of the fields. Additionally, the constraints transform into algebraic expressions in terms of the transforms of the fields, thus acquiring an algebraic look. Regardless of whether these seemingly algebraic conditions are practical boundary conditions for a numerical simulation, the argument does not hold up in the non-linear case.

The idea of imposing the vanishing of the incoming constraints as boundary conditions is pursued further in [9], where space derivatives of the fields are eliminated in favor of time derivatives in the expression of the incoming constraints in terms of the fundamental variables. In this case, a different formulation of the Einstein equations is used [10], restricted to spherical symmetry.

Clearly not all formulations have the problem of imposing the constraints at the boundary. Suppose there is a formulation of the Einstein equations that has no incoming constraints, that is: a formulation in which the constraints propagate upwards along the boundary. In such a case, the constraints will be satisfied at the boundary by virtue of the initial values alone, and there is no need to impose additional conditions on the boundary values in order to enforce them. The boundary values of the incoming fields in this case must be arbitrary. Such formulations exist at least in symmetry-reduced cases, and some boundary problems for those formulations have been studied [11].

Here we stray away from the general trend of seeking a way to impose the constraints along the boundary, in order to propose a seemingly unrelated method to write down equations that must hold among the boundary values of many quite generic first-order formulations of the Einstein equations. These consist of the vanishing of the four components of the projection of the Einstein tensor along the normal to the boundary.

In Section 2 we briefly describe the formulation of the Einstein equations that we choose to write down the proposed boundary conditions explicitly in a model case. The discussion in Section 2 hinges heavily on the existence of a complete set of characteristic fields, which is guaranteed by the strong hyperbolicity of the formulation chosen (as opposed to weak hyperbolicity). In Section 3 we show that such boundary conditions are consistent with constraint propagation, realize the goal of guaranteeing the vanishing of the incoming constraints at the boundary and, furthermore, they coincide with the boundary conditions proposed in [9] by the trading of space and time derivatives alluded to above. The discussion on Section 3 hinges heavily on the existence of a complete set of characteristic constraint fields, namely: on the fact that the constraint propagation is strongly hyperbolic (as opposed to weakly hyperbolic). In Section 4 we state the generality of the arguments with regards to three-dimensional strongly hyperbolic formulations, namely: the aspects of the argument that apply to any strongly hyperbolic formulation of the Einstein equations irrespective of particularities such as the specific characteristic fields and speeds. Formulation-dependent features such as any explicit form of the boundary conditions arising from the projection of the Einstein equations perpendicularly to a boundary are intentionally excluded from Section 4 with the purpose of exposing the true reach and relevance of the argument across the lines delimiting alternative strongly hyperbolic formulations. Concluding remarks appear in Section 5.

2. Boundary conditions for the Einstein-Christoffel formulation with spherical symmetry

As usual, in a foliation of spacetime by level surfaces of a time function t, and using t as a coordinate, the spacetime metric can be viewed as a time dependent Riemannian metric evolving in time according to a freely specifiable rate and labeling of the 3-space. From now on, we use the term *metric* with no qualifier to refer to the evolving Riemannian metric of the slices. In this section we use the Einstein-Christoffel (EC) formulation [10], restricted to spherical symmetry. The full 3-D equations and their structure of characteristics can be found in [12], as well as the restriction to spherical symmetry. We borrow the notation directly from [12]. With the restriction to spherical symmetry, the spacetime metric has the form

$$ds^2 = -N^2 dt^2 + g_{rr}(dr + \beta^r dt)^2 + g_T r^2 (d\theta^2 + \sin^2\theta d\phi^2) \tag{1}$$

where g_{rr}, g_T, β^r and N are functions only of r and t. The shift β^r and the densitized lapse

$$\widetilde{\alpha} \equiv \frac{N}{g_T\sqrt{g_{rr}}} \tag{2}$$

are considered arbitrarily given. The EC formulation of the Einstein equations restricted to spherical symmetry consists of six fundamental variables $(g_{rr}, g_T, K_{rr}, K_T, f_{rrr}, f_{rT})$ which evolve according to six evolution equations and whose initial data satisfies four constraints [12]. The evolution equations are

$$\partial_t g_{rr} - \beta^r \partial_r g_{rr} = -2NK_{rr} + 2g_{rr}\partial_r\beta^r, \tag{3a}$$

$$\partial_t g_T - \beta^r \partial_r g_T = -2NK_T + 2\frac{\beta^r}{r} g_T, \tag{3b}$$

$$\begin{aligned}\partial_t K_{rr} - \beta^r\partial_r K_{rr} + \frac{N}{g_{rr}}\partial_r f_{rrr} = N\Bigg[&2f^r{}_{rr}\left(f^r{}_{rr} + \frac{1}{r} - \frac{4f_{rT}}{g_T}\right) + K_{rr}\left(2\frac{K_T}{g_T} - K^r_r\right) \\ &-\frac{6}{r^2} - 6\left(\frac{f_{rT}}{g_T}\right)^2 - \partial_r^2 \ln\widetilde{\alpha} - (\partial_r \ln\widetilde{\alpha})^2 \\ &+ \left(\frac{4}{r} - f^r{}_{rr}\right)\partial_r\ln\widetilde{\alpha}\Bigg] + 2K_{rr}\partial_r\beta_r \\ &+ 4\pi N(Tg_{rr} - 2S_{rr}),\end{aligned} \tag{3c}$$

$$\begin{aligned}\partial_t K_T - \beta^r\partial_r K_T + \frac{N}{g_{rr}}\partial_r f_{rT} &= N\left(K_T K^r_r + \frac{1}{r^2} - \frac{2f_{rT}^2}{g_{rr}g_T} - \frac{f_{rT}}{g_{rr}}\partial_r\ln\widetilde{\alpha}\right) \\ &+ \frac{2\beta^r}{r}K_T,\end{aligned} \tag{3d}$$

$$\begin{aligned}\partial_t f_{rrr} - \beta^r\partial_r f_{rrr} + \frac{N}{g_{rr}}\partial_r K_{rr} = N\Bigg[&4g_{rr}\frac{K_T}{g_T}\left(3\frac{f_{rT}}{g_T} - f^r{}_{rr} + \frac{2}{r} - \partial_r\ln\widetilde{\alpha}\right) \\ &-K_{rr}\left(10\frac{f_{rT}}{g_T} + f^r{}_{rr} - \frac{2}{r} + \partial_r\ln\widetilde{\alpha}\right)\Bigg] \\ &+ 3f_{rrr}\partial_r\beta^r + g_{rr}\partial_r^2\beta^r + 16\pi NJ_r g_{rr},\end{aligned} \tag{3e}$$

$$\begin{aligned}\partial_t f_{rT} - \beta^r\partial_r f_{rT} + \frac{N}{g_{rr}}\partial_r K_T &= NK_T\left(2\frac{f_{rT}}{g_T} - f^r{}_{rr} - \partial_r\ln\widetilde{\alpha}\right) \\ &+ \left(\partial_r\beta^r + \frac{2\beta^r}{r}\right)f_{rT}.\end{aligned} \tag{3f}$$

The constraints are

$$\mathcal{C} \equiv \frac{\partial_r f_{rT}}{g_{rr} g_T} - \frac{1}{2r^2 g_t} + \frac{f_{rT}}{g_{rr} g_T}\left(\frac{2}{r} + \frac{7 f_{rT}}{2g_T} - f^r{}_{rr}\right) - \frac{K_T}{g_T}\left(K^r_r + \frac{K_T}{2g_T}\right) + 4\pi\rho = 0, \quad (4a)$$

$$\mathcal{C}_r \equiv \frac{\partial_r K_T}{g_T} + \frac{2K_T}{r g_T} + \frac{f_{rT}}{g_T}\left(K^r_r + \frac{K_T}{g_T}\right) + 4\pi J_r = 0, \quad (4b)$$

$$\mathcal{C}_{rrr} \equiv \partial_r g_{rr} + \frac{8 g_{rr} f_{rT}}{g_T} - 2f^r{}_{rr} = 0, \quad (4c)$$

$$\mathcal{C}_{rT} \equiv \partial_r g_T + \frac{2g_T}{r} - 2f_{rT} = 0. \quad (4d)$$

Here $\mathcal{C}$ and $\mathcal{C}_r$ are the scalar and vector constraints, respectively, whereas the vanishing of $\mathcal{C}_{rrr}$ and $\mathcal{C}_{rT}$ defines the variables f_{rrr} and f_{rT}. We are here interested only in the vacuum equations, so we set all the source terms to zero ($T = S_{rr} = J_r = \rho = 0$), and we do not need to consider matter equations.

In essence, the evolution system is strongly hyperbolic because one can find a complete set of linearly independent combinations of the fundamental variables such that the principal part of the equations decouples. Such combinations are referred to as the characteristic variables. In a linear homogeneous system, the characteristic variables propagate their initial values exactly along the characteristic lines, and may be interpreted as waves propagating with the characteristic speed. In our case [12], the characteristic fields and their characteristic speeds are

$$U^0_r \equiv g_{rr} \qquad (v_c = \beta^r) \quad (5a)$$

$$U^0_T \equiv g_T \qquad (v_c = \beta^r) \quad (5b)$$

$$U^\pm_r \equiv K_{rr} \pm \frac{f_{rrr}}{\sqrt{g_{rr}}} \qquad (v_c = \beta^r \mp \widetilde{\alpha} g_T) \quad (5c)$$

$$U^\pm_T \equiv K_T \pm \frac{f_{rT}}{\sqrt{g_{rr}}} \qquad (v_c = \beta^r \mp \widetilde{\alpha} g_T) \quad (5d)$$

Here we have changed the signs of the characteristic speeds relative to [12] in order to agree with the sign conventions in [5]. The characteristic fields give insights into how much data one can or must provide in addition to the initial data, depending on the region of spacetime where the solution is sought. If the region of interest has a boundary, in general one may need to prescribe additional data on this boundary in order to obtain a unique solution within that region. The additional data are the values of the characteristic fields that enter the region from outside. If such values are not prescribed, then there are many solutions for the same initial data in the region of interest. For each boundary value prescribed arbitrarily – as long as it is consistent with the initial data at the intersection of the boundary with the initial slice –, there is a unique solution. No boundary value may be freely prescribed for the characteristic fields that cross the boundary from the inside, or that run along the boundary, since such values are determined by the initial data already, in principle. In practice, since the equations are not linear nor homogeneous, the actual values of the outgoing characteristic fields at the boundary are not known in advance. Most of the time the values of some or even all the fields at the boundary, regardless of formulation type, are prescribed by means of some type of wave condition, as in [13] and, more elaborately, in [14].

In our problem, we restrict attention now to a region inside a fixed value of r. There is a set of incoming characteristic fields, which are the ones that have positive

characteristic speeds in our convention, which we will be able to single out once we choose the values of the densitized lapse and shift. Suppose initial data are prescribed with vanishing values of the constraints. One can now prescribe the values of the incoming fields independently of the initial values. The question is: are such values free? It is worth investigating the possibility that the Einstein equations themselves may restrict the boundary values.

Consider the following argument. The boundary is a surface of constant radius in spacetime. The unit normal vector to the boundary is well defined. We denote it by $e^a = (e^t, e^r, e^\theta, e^\phi)$ and it can easily be computed from the gradient vector $r_{,a} = (0, 1, 0, 0)$ by $e^a = {}^4g^{ab} r_{,b} / \sqrt{{}^4g^{ab} r_{,b}\, r_{,a}} = {}^4g^{ar} / \sqrt{{}^4g^{rr}}$, where ${}^4g^{ab}$ is the contravariant spacetime metric. In our case, we have

$$e^t = \frac{\beta^r}{\widetilde{\alpha} g_T \sqrt{g_{rr}(\widetilde{\alpha}^2 g_T^2 - (\beta^r)^2)}}, \tag{6a}$$

$$e^r = \frac{\sqrt{\widetilde{\alpha}^2 g_T^2 - (\beta^r)^2}}{\widetilde{\alpha} g_T \sqrt{g_{rr}}}, \tag{6b}$$

$$e^\theta = e^\phi = 0. \tag{6c}$$

The projector p_{ab} on the boundary surface can be found from the spacetime metric and the unit normal to the surface via

$$p_{ab} \equiv {}^4g_{ab} - e_a e_b \tag{7}$$

with latin indices $a, b, c\ldots$ raised and lowered with the spacetime metric. In particular, we are interested in the projector with mixed indices

$$p^a{}_b = \delta^a{}_b - e^a e_b \tag{8}$$

We saturate the indices of the Einstein equations $G_{ab} = 0$ in two different ways. One equation is obtained by contracting twice with the normal to the boundary surface, e^a. We have

$$G_{ab} e^a e^b = G_{tt}(e^t)^2 + 2G_{tr} e^r e^t + G_{rr}(e^r)^2 = 0 \tag{9}$$

Another set of equations is obtained by contracting one idex with e^a and the other index with $p^b{}_c$, namely: $G_{ab} e^a p^b{}_c = 0$. But because of the spherical symmetry, this contraction is identically zero for $c = \theta, \phi$. The remaining equations are

$$G_{ab} e^a p^b{}_t = G_{tt} e^t + G_{tr} e^r = 0, \tag{10}$$

$$G_{ab} e^a p^b{}_r = (G_{tt} e^t + G_{tr} e^r) p^t{}_r + (G_{tr} e^t + G_{rr} e^r) p^r{}_r = 0. \tag{11}$$

Clearly, if (10) and (9) are satisfied, then (11) is an identity. Thus there are only two independent equations, and we pick (10) and (9). These equations are extremely valuable for the boundary value problem because they contain no second-order derivatives with respect to r irrespectively of the choice of β^r or $\widetilde{\alpha}$. In the case of a first-order formulation, like ours, such equations are said to be *interior to the boundary surface* – not to confuse with the *interior* of our *region!* In the following, we restrict ourselves to $\beta^r = 0$ and $\widetilde{\alpha} = 1$ just for the sake of argument. Non trivial choices of densitized lapse and shift only make the argument more complicated without introducing any real obstacle.

Explicitly, for the case of $\beta^r = 0$ and $\widetilde{\alpha} = 1$, we have

$$G_{ab} e^a e_b = 0 \;=\; -\frac{1}{r^2 {g_T}^2 {g_{rr}}^2}\Big(2r^2 {g_{rr}}^{3/2} \dot{K}_T - 3r^2 g_{rr} {f_{rT}}^2 - 4r g_T g_{rr} f_{rT}$$

$$+ 2r^2 g_T f_{rT} f_{rrr} - {g_{rr}}^2 g_T + r^2 g_{rr} {K_T}^2\Big), \tag{12a}$$

$$G_{ab}e^a p^b{}_t = 0 = \frac{2}{r g_T {g_{rr}}^{3/2}}\Big(r g_{rr}\dot{f}_{rT} + r g_T \sqrt{g_{rr}} f_{rT} K_{rr} - \sqrt{g_{rr}} K_T \left(r g_{rr} f_{rT}\right.$$
$$\left. + 2 g_T g_{rr} - r g_T f_{rrr}\right)\Big). \tag{12b}$$

We can put these equations entirely in terms of the characteristic fields, using

$$K_T = \frac{1}{2}\left(U_T^+ + U_T^-\right), \tag{13a}$$

$$f_{rT} = \frac{\sqrt{g_{rr}}}{2}\left(U_T^+ - U_T^-\right), \tag{13b}$$

$$K_{rr} = \frac{1}{2}\left(U_r^+ + U_r^-\right), \tag{13c}$$

$$f_{rrr} = \frac{\sqrt{g_{rr}}}{2}\left(U_r^+ - U_r^-\right). \tag{13d}$$

and the fact that the metric components g_{rr} and g_T are characteristic fields themselves. For the derivatives $\dot{K}_T$ and $\dot{f}_{rT}$ we take derivatives of (13a) and (13b) and use (3a) – written in terms of the characteristic fields– to substitute $\dot{g}_{rr}$ which will appear in $\dot{f}_{rT}$. Explicitly, for $\beta^r = 0$ and $\tilde{\alpha} = 1$ we have

$$\dot{K}_T = \frac{1}{2}\left(\dot{U}_T^+ + \dot{U}_T^-\right) \tag{14a}$$

$$\dot{f}_{rT} = \frac{\sqrt{g_{rr}}}{2}\left(\dot{U}_T^+ - \dot{U}_T^-\right) - \frac{g_T}{4}\left(U_r^+ + U_r^-\right)\left(U_T^+ - U_T^-\right) \tag{14b}$$

On account of (14a-14b), one can see by inspection that when both equations (12a-12b) are written out entirely in terms of characteristic fields, they involve the time derivatives only of $U_T^\pm$, but not of $U_r^\pm$. Furthermore, $\dot{U}_T^\pm$ appear linearly in both equations. We can algebraically solve the two equations for both $\dot{U}_T^\pm$ as independent variables in terms of the rest. We readily find:

$$\dot{U}_T^- = \frac{1}{2r^2\sqrt{g_{rr}}}\Big(g_{rr} g_T + r^2 g_{rr}\left(U_T^-\right)^2 - 2r^2 g_{rr} U_T^- U_T^+$$
$$- 4r\sqrt{g_{rr}} g_T U_T^- + r^2 g_T U_T^- U_r^+ - r^2 g_T U_T^- U_r^-\Big) \tag{15}$$

and

$$\dot{U}_T^+ = \frac{1}{2r^2\sqrt{g_{rr}}}\Big(g_{rr} g_T + r^2 g_{rr}\left(U_T^+\right)^2 - 2r^2 g_{rr} U_T^+ U_T^-$$
$$+ 4r\sqrt{g_{rr}} g_T U_T^+ - r^2 g_T U_T^+ U_r^+ + r^2 g_T U_T^+ U_r^-\Big) \tag{16}$$

Both equations (15) and (16) are necessary because they are linearly independent. The question is: how are we to interpret them within the framework of the initial-boundary value problem?

To start with, the fields g_{rr} and g_T may be considered as known sources, because they travel upwards along the boundary. If the problem was linear and homogeneous, then the values of g_{rr} and g_T at any point on the boundary would be exactly their initial values. Because the problem is non-linear, the values at any point on the boundary must be integrated, but can still be regarded as sources. A similar thinking may be used for the outgoing characteristic field U_r^+, with the complication that U_r^+ travels towards the boundary but not directly upwards, and thus the computation of its values at the boundary will be more involved, but conceptually no different.

Additionally, we may take the point of view that the incoming characteristic field U_r^- is arbitrary, since there are only two equations to satisfy at the boundary and $\dot{U}_r^-$ does not appear in any of them. From this perspective, the value of this field at the boundary is truly free, and must be considered as one true degree of freedom of the boundary value problem for the Einstein equations, in addition to the degrees of freedom contained in the initial data.

Continuing within such an interpretation, Eq. (15) provides exact boundary values for the incoming characteristic field U_T^- in terms of the free incoming field U_r^- and the fields that either run along the boundary or cross from the inside, all of which are determined by the initial data in principle, as argued before.

There remains Eq. (16). From our point of view, this equation predicts boundary values for U_T^+. There is no doubt that it must hold, in order for the solution that we are seeking to satisfy all the Einstein equations at all points on the boundary surface. But U_T^+ is, in principle, determined by the initial data by propagation along characteristics, as explained before. There seem to be two possibilities here: either the outgoing field U_T^+ propagated from the initial data satisfies Eq. (16), or it does not. If the boundary values of U_T^+ do not satisfy Eq. (16), then the initial-boundary value problem is inconsistent and *can not* yield a solution to the Einstein equations in the region of interest.

Alternatively, if the boundary values of U_T^+ propagated along characteristics do satisfy Eq. (16), then the initial-boundary value problem is consistent. In this case, Eq. (16) can be used to prescribe the values of U_T^+ with more accuracy and reliability than the propagation along characteristics discussed above because it is an ordinary differential equation, and should be preferred.

In the following Section we show that the initial-boundary value problem is consistent and we should be using Eq. (16) to prescribe the boundary values of U_T^+ instead of propagating them by characteristics from the initial data, because the propagation by characteristics is inaccurate in itself –even more so in the case that the characteristic speeds depend on the fields themselves and must be calculated at every step. One can anticipate that this is the case based on some intuition. Why should the boundary values of the outgoing fields be "constrained" by an equation such as Eq. (16)? Because, in principle, the boundary values of the outgoing fields must reflect in some way the constraints on the intial data that determine them. The initial data are related; their inter-relationships must be propagated along characteristics. Eq. (16) may well be a shortcut out of constraint propagation.

3. Consistency with constraint propagation

If thought of as functions which could take any real values, the initial constraints $\mathcal{C}, \mathcal{C}_r, \mathcal{C}_{rrr}, \mathcal{C}_{rT}$ evolve in time according to another strongly hyperbolic evolution system, with the following characteristic fields and speeds[9]:

$$C_1 = \mathcal{C} + \frac{\mathcal{C}_r}{\sqrt{g_{rr}}} \qquad (v_1^c = \beta^r - \widetilde{\alpha} g_T) \tag{17a}$$

$$C_2 = \mathcal{C} - \frac{\mathcal{C}_r}{\sqrt{g_{rr}}} \qquad (v_2^c = \beta^r + \widetilde{\alpha} g_T) \tag{17b}$$

$$C_3 = \mathcal{C}_{rrr} \qquad (v_3^c = \beta^r) \tag{17c}$$

$$C_4 = \mathcal{C}_{rT} \qquad (v_4^c = \beta^r) \tag{17d}$$

In the case of $\beta^r = 0$ and $\widetilde{\alpha} = 1$, C_2 is the only incoming characteristic field. If one wishes for all four functions to take the value zero in the region interior to some fixed radius, then three of them will be zero by choice of initial data, but C_2 must be set to zero at the boundary in order to propagate inwardly from there. But one simply may set the value of C_2 to zero because one does not evolve the constraint propagation system. It is one thing to impose the value 0 on the function C_2, but a different thing to expect C_2 to be vanishing at the boundary as a function of the fundamental variables of evolution. In particular, by inspection one can readily see that C_2 contains the derivative of U_T^- with respect to r. Therefore, although it is natural and desirable to have $C_2 = 0$ on the boundary, it is an impossible task to have $C_2 = 0$ as a boundary condition for the fundamental fields of evolution. C_2 will have to vanish for the evolution to produce a solution of the Einstein equations, and thus the vanishing of C_2 and all the other constraints must be checked for consistency at the boundary, after the solution has been found and the $r-$derivatives can be evaluated; but it is not practical as a boundary condition.

A strategy that has been proposed [9] to circumvent this obstacle is to use the evolution equations to turn the $r-$derivatives of the characteristic fields that appear in the expression of C_2 into $t-$derivatives. The procedure must work in this symmetry-reduced case because, by construction, all characteristic fields have time derivatives proportional to their $r-$derivatives, up to terms of zeroth order. Therefore, $C_2 = 0$ can be turned into an evolution equation for U_T^- that will be restricted to the boundary. Following [9], we do that next, maintaining the restrictions $\beta^r = 0$ and $\widetilde{\alpha} = 1$. We have explicitly

$$\begin{aligned} C_2 = & -\frac{1}{2r^2 {g_T}^2 {g_{rr}}^{5/2}} \Big(2r^2 g_T g_{rr} \partial_r K_T - 2r^2 g_T {g_{rr}}^{3/2} \partial_r f_{rT} + g_T {g_{rr}}^{5/2} \\ & - 4r g_T {g_{rr}}^{3/2} f_{rT} - 7r^2 {g_{rr}}^{3/2} f_{rT} + 2r^2 g_T \sqrt{g_{rr}} f_{rT} f_{rrr} + 2r^2 g_T {g_{rr}}^{3/2} K_T K_{rr} \\ & + r^2 {g_{rr}}^{5/2} {K_T}^2 + 4r g_t g_{rr}^2 K_T - 2r^2 g_T g_{rr} f_{rT} K_{rr} - 2r^2 g_{rr}^2 f_{rT} K_T \Big) \end{aligned} \tag{18}$$

Solving for $\partial_r f_{rT}$ from the evolution equation (3*d*), and for $\partial_r K_T$ from the evolution equation (3*f*), substituting into (18), and replacing all appearances of f_{rT}, f_{rrr}, K_T and K_{rr} in terms of the characteristic fields using (13*a*-13*d*), Eq. (18) turns into a new expression, which is not an initial constraint anymore, so we use quotes to make this point explicit:

$$\begin{aligned} \text{“}C_2\text{”} = & -\frac{1}{2r^2 {g_T}^2 {g_{rr}}^{5/2}} \Big(2r^2 {g_{rr}}^2 \dot{U}_T^- - r^2 {g_{rr}}^{5/2} (U_T^-)^2 - r^2 g_T {g_{rr}}^{3/2} U_T^- U_T^+ \\ & + r^2 g_T {g_{rr}}^{3/2} U_T^- U_r^- - g_T {g_{rr}}^{5/2} + 2r^2 {g_{rr}}^{5/2} U_T^+ U_T^- + 4r g_T {g_{rr}}^2 U_T^- \Big) \end{aligned} \tag{19}$$

Setting “C_2” $= 0$ yields a boundary condition for U_T^-. The point is that $C_2 = 0$ on the boundary is not the same as “C_2” $= 0$, but they are the same along the boundary if evaluated on fundamental fields that satisfy the evolution equations. In other words: they differ by a linear combination of the evolution equations. But more interestingly, “C_2” $= 0$ is exactly the same equation as (15), as can be verified by inspection. Therefore the method of “trading” space derivatives for time derivatives advocated in [9] is equivalent to solving one of the Einstein equations that are *interior to the boundary,* there being a two-dimensional set of such equations. This could be expected because the projections of the Einstein equations along the direction normal to a surface of fixed radius are the linear combinations that have no second-order $r-$derivatives, and clearly the “trading” is equivalent to linearly combining the

Einstein equations. Now, because Eq. (16) is equivalent to $C_2 = 0$ on the boundary, it guarantees that C_2 will vanish in the interior by propagation along characteristics. So Eq. (16) is not only consistent with the boundary value problem, but is necessary as well.

We can also use the "trading" of space derivatives for time derivatives to write an equation "C_1" $= 0$ from $C_1 = 0$ along the boundary. This is exactly Eq. (16), the meaning of which is explained in the previous section, and we conclude that Eq. (16) and C_1 are equivalent along the boundary if the evolution equations are satisfied. But how much of this argument proves that the constraint has been "propagated" and where does the propagation start? Clearly C_1 propagates out from the initial data towards the boundary, therefore C_1 will vanish along the boundary as a consequence of its vanishing initially. Since the evolution will be satisfied by construction, then "C_1" will vanish as a consequence of C_1 vanishing initially as well. Seemingly, thus, the boundary values of the fields satisfying Eq. (16) must be consistent with the initial values satisfying $C_1 = 0$. For this reason, and because Eq. (16) is an ordinary differential equation, it is not only consistent but advisable to use Eq. (16) to obtain the values of U_T^+ instead of propagating them from the initial values along characteristics. At any rate, Eq. (16) should be checked for consistency regardless of method.

Two objections may come to mind on a surface glance to Eq. (16), which we want to anticipate. It might conceivably be argued that if the values of U_T^+ at the boundary are obtained from the initial values by propagation along characteristics, then Eq. (16) could also be thought of as an equation for the incoming field U_r^- with U_T^+ given, instead of prescribing U_T^+ with a free U_r^-. But this is a flawed line of thought, for the boundary values of U_T^+ propagated along characteristics are consistent with Eq. (16), and substituting U_T^+ into Eq. (16) must yield an identity, leaving no equation to solve for U_r^-, which would thus remain arbitrary.

Secondly, how is it that Eq. (16) involving a free boundary field U_r^- can be consistent with propagation of U_T^+ along characteristics, given that U_r^- is a field which the initial data know nothing about? The initial data do not know about U_r^-, but the characteristics do, because the problem is non linear. Thus the propagation of U_T^+ along characteristics will necessarily yield boundary values of U_T^+ that know about earlier boundary values of incoming fields. It is fine for Eq. (16) to involve U_r^-, as long as it does not involve $\dot{U}_r^-$.

For additional support of our scheme to use Eq. (15) and Eq. (16) to prescribe U_T^- and U_T^+, respectively, assuming that U_r^- is given freely, we can linearize the equations around flat space and verify that the scheme is consistent, since the boundary value problem of linear hyperbolic equations is clear and enlightening. We have

$$g_{rr} = 1 + \widehat{g}_{rr} \tag{20a}$$

$$g_T = 1 + \widehat{g}_T \tag{20b}$$

$$K_{rr} = \widehat{K}_{rr} \tag{20c}$$

$$K_T = \widehat{K}_T \tag{20d}$$

$$f_{rrr} = \frac{4}{r} + \widehat{f}_{rrr} \tag{20e}$$

$$f_{rT} = \frac{1}{r} + \widehat{f}_{rT} \tag{20f}$$

where all hatted quantities are small. We will keep only linear terms in such quantities.

The evolution equations become

$$\dot{\widehat{g}}_{rr} = -2\widehat{K}_{rr} \tag{21a}$$

$$\dot{\widehat{g}}_{T} = -2\widehat{K}_{T} \tag{21b}$$

$$\dot{\widehat{K}}_{rr} + \partial_r \widehat{f}_{rrr} = -\frac{42}{r^2}\widehat{g}_{rr} + \frac{48}{r^2}\widehat{g}_T + \frac{10}{r}\widehat{f}_{rrr} - \frac{44}{r}\widehat{f}_{rT}, \tag{21c}$$

$$\dot{\widehat{K}}_{T} + \partial_r \widehat{f}_{rT} = \frac{1}{r^2}\widehat{g}_{rr} + \frac{2}{r^2}\widehat{g}_T - \frac{4}{r}\widehat{f}_{rT} \tag{21d}$$

$$\dot{\widehat{f}}_{rrr} + \partial_r \widehat{K}_{rr} = -\frac{12}{r}\widehat{K}_{rr} + \frac{4}{r}\widehat{K}_T \tag{21e}$$

$$\dot{\widehat{f}}_{rT} + \partial_r \widehat{K}_{T} = -\frac{2}{r}\widehat{K}_T \tag{21f}$$

Thus the characteristic fields are $\widehat{g}_{rr}, \widehat{g}_T$ and

$$\widehat{U}_T^{\pm} \equiv \widehat{K}_T \pm \widehat{f}_{rT}, \tag{22a}$$

$$\widehat{U}_r^{\pm} \equiv \widehat{K}_{rr} \pm \widehat{f}_{rrr}. \tag{22b}$$

Our boundary conditions, Eqs. (15) and (16), linearize to

$$\dot{\widehat{U}}_T^{-} = \frac{9}{2r^2}\widehat{g}_{rr} - \frac{3}{2r^2}\widehat{g}_T + \frac{1}{r}\widehat{U}_r^{-} + \frac{1}{r}\widehat{U}_T^{+}\ , \tag{23}$$

and

$$\dot{\widehat{U}}_T^{+} = \frac{9}{2r^2}\widehat{g}_{rr} - \frac{3}{2r^2}\widehat{g}_T - \frac{1}{r}\widehat{U}_r^{+} - \frac{1}{r}\widehat{U}_T^{-}\ , \tag{24}$$

respectively. Thus (23) can be used to prescribe values for $\widehat{U}_T^-$ if $\widehat{U}_r^-$ is given arbitrarily, whereas (24) can be used to calculate values for $\widehat{U}_T^+$ irrespective of $\widehat{U}_r^-$, as anticipated.

4. Boundary conditions for generic three-dimensional strongly hyperbolic formulations

How much of the argument in spherical symmetry actually depends on the symmetry assumption? Not a great deal. Suppose we have a strongly hyperbolic formulation of the Einstein equations in terms of 6+6+18 variables representing the three-metric and all its first derivatives. The evolution equations then look like

$$\dot{u} = A^i \partial_i u + b \tag{25}$$

where u is the 30-dimensional vector of all the fundamental variables, and there are 4+18 constraints on the initial data:

$$\mathcal{C} = 0 \tag{26a}$$

$$\mathcal{C}_i = 0 \tag{26b}$$

$$\mathcal{C}_{ijk} = 0 \tag{26c}$$

where $\mathcal{C}$ and $\mathcal{C}_i$ are the scalar and vector constraint, respectively, and $\mathcal{C}_{ijk}$ are the constraints necessary to reduce the equations from second to first order in space (they define the 18 first order variables). Wherever the set of evolution equations and the constraints are satisfied, the ten Einstein equations $G_{ab} = 0$ for the ten components of the spacetime metric g_{ab} are satisfied, equivalently.

Suppose the evolution equations (25) are strongly hyperbolic, and that they imply, in the usual manner [15], a second (or *subsidiary*) system of equations for the constraint functions $\mathcal{C}, \mathcal{C}_i$ and $\mathcal{C}_{ijk}$. Suppose that this second system of equations, for the

constraints, is also strongly hyperbolic. Assume we have identified the characteristic variables of both strongly hyperbolic systems.

Suppose now that we seek a solution in the region interior to some fixed value of the coordinate x^1 (any spacelike coordinate). The unit normal to the boundary is then

$$e^a = \frac{g^{ab}\delta^1_b}{\sqrt{g^{ab}\delta^1_a\delta^1_b}} = \frac{g^{1a}}{\sqrt{g^{11}}} \tag{27}$$

and the projector on the boundary surface is $p^a{}_b = \delta^a_b + e^a e_b$ with $e_b = \delta^1_b/\sqrt{g^{11}}$. We can write down $G_{ab}e^a e^b = 0$ and $G_a b e^a p^b{}_c = 0$. In general these will be four equations with *no second derivatives with respect to* x^1, as can be proven by direct calculation. Contracting the Einstein tensor with e^a yields a vector equation with two pieces:

$$G_{ab}e^a = R_{ab}e^a - \frac{\delta^1_b}{2\sqrt{g^{11}}}R. \tag{28}$$

We next show that the components $b \neq 1$ of $R_{ab}e^a$ have no second derivatives of the metric with respect to x^1, and that the Ricci scalar term cancels out the second x^1-derivatives that appear in the first term for $b = 1$. We use the following well-known expression of the Ricci tensor [16] in which the second-order derivatives of the metric are explicit:

$$R_{ab} = \frac{1}{2}g^{cd}\left(g_{cb,ad} + g_{ad,cb} - g_{cd,ab} - g_{ab,cd}\right) + g^{cd}\left(\Gamma^e_{ad}\Gamma_{edb} - \Gamma^e_{ab}\Gamma_{ecd}\right). \tag{29}$$

Contracting with e^a yields

$$R_{ab}e^a = \frac{1}{2\sqrt{g^{11}}}\left(g^{1a}g^{1c}g_{ac,1b} - g^{11}g^{cd}g_{cd,1b}\right) + \ldots \tag{30}$$

where $\ldots$ represents terms that have no second derivatives with respect to x^1. Thus $R_{ab}e^a$ has no second x^1-derivatives except for $b = 1$. The second part of the Einstein tensor is nonvanishing only for $b = 1$. Therefore the only component of $G_{ab}e^a$ that might contain a second x^1-derivative is $b = 1$. Now R is exactly

$$R = g^{1a}g^{1c}g_{ac,11} - g^{11}g^{cd}g_{cd,11} + \ldots \tag{31}$$

Therefore, the terms with second x^1-derivatives of the Ricci scalar part of the Einstein tensor cancel exactly with those in the Ricci tensor part, with the consequence that $G_{ab}e^a$ has no second x^1-derivatives of the metric for any value of b. Contracting with e^b or with $p^b{}_c$ will not add new second x^1-derivatives, but provides perhaps a convenient splitting of the four equations. The four equations

$$\mathcal{B}_b \equiv G_{ab}e^a = 0 \tag{32}$$

or equivalently

$$G_{ab}e^a e^b = 0 \quad \text{and} \quad G_{ab}e^a p^b{}_c = 0 \tag{33}$$

must be satisfied by any solution to the Einstein equations and are *interior* to the boundary surface, which makes them ideal boundary conditions. There are four equations and thirty variables, but of the thirty variables many are outgoing or run along the boundary. In fact, one can always write the evolution system so that the metric components propagate with vanishing characteristic speeds, regardless of the choice of lapse and shift. So we have at most $(30-6)/2$ relevant incoming characteristic fields. Clearly, some of the incoming fields will be left arbitrary at the boundary – perhaps more than 8, since in our example in spherical symmetry we find that one of

the boundary conditions applies to an outgoing field instead of an incoming one. The particulars of which characteristic fields will be affected by the boundary equations will depend entirely on the details of the first-order strongly hyperbolic formulation of choice. As an illustration, the direct generalization of our argument to the case of the Einstein-Christoffel formulation in three-dimensions without the restriction of spherical symmetry will be reported elsewhere.

Conceptually, the scheme runs as follows. Choose initial data that satisfy (4). Choose boundary data that satisfy (32). The vanishing of the initial constraints guarantees that the outgoing and static constraints will remain vanishing in the region of interest. The boundary equations (32) are equivalent to the vanishing of the incoming constraints at the boundary on account of their being related to the incoming constraints by linear combinations with the Einstein equations that *are* satisfied at the boundary – the evolution is satisfied by construction and the outgoing constraints are satisfied by propagation along characteristics. Therefore the boundary equations (32) guarantee that the incoming constraints are vanishing at the boundary, which in turn guarantees that they will remain vanishing in the region of interest.

Although the preceding calculation assumes that the normal to the surface of fixed value of x^1 is spacelike, clearly Eqs. (32) will be interior to the surface of fixed value of x^1 even if the normal is timelike, in which case the proof that the equations contain no second x^1-derivatives runs the same if we substitute $\sqrt{g^{11}}$ with $\sqrt{|g^{11}|}$. This remark applies to interior boundaries that lie within the event horizon of a black hole spacetime, and may be relevant to numerical simulations of binary black holes.

On the other hand, in this Section we have restricted the discussion to three-dimensional boundaries taken one at a time. As is the case with any three-dimensional boundary-value problem, the intersection of two boundaries requires special attention. In this respect, consistency issues between the boundary equations that may arise at the intersection of two boundary surfaces remain to be studied, and their resolution may depend on the particulars of the formulation of the initial value problem at hand. The interested reader is referred to [17] where a particular representation of the projection of the Einstein equations on the boundary is implemented and corners and edges are treated, for the case of an ADM-like formulation of the initial value problem.

5. Concluding remarks and outlook

Even though we have, for the most part, developed our arguments explicitly in the case of spherical symmetry for the EC formulation, the fundamental relevance of the argument to the initial-boundary value problem of the Einstein equations does not depend on the symmetry restrictions nor on the particulars of the hyperbolic formulation, as we show in Section 4.

We argue that given any boundary at a fixed value of a coordinate of an initial-boundary value problem for the Einstein equations, the vanishing of the components of the projection of the Einstein tensor G_{ab} along the normal e^a to the boundary, namely $G_{ab}e^b \equiv \mathcal{B}_a = 0$, constitute necessary and consistent boundary conditions, for essential reasons, as follows.

First, the components of the projection of the Einstein tensor along the normal to such a boundary, $\mathcal{B}_a$, contain no second-order derivatives sticking out of the boundary. If the initial value problem is stated in first order form, such as any of the hyperbolic formulations available, their vanishing becomes differential equations for the boundary values of the fundamental fields excluding their derivatives across

the boundary. Moreover, if the initial value problem is stated in second-order form in the space coordinates, such as ADM form [18, 19] or conformal form [13, 20], they constitute a type of mixed Neumann-Dirichlet conditions, the appropriateness of which is worth investigating in full depth. In this regard, some representations of $\mathcal{B}_a = 0$ as boundary conditions have been used along with an ADM type formulation of the linearized Einstein equations in [17] to investigate the consistency of pure Neumann or Dirichlet conditions with the evolution.

Second, in the case of strongly hyperbolic formulations that propagate the constraints in a strongly hyperbolic fashion, the vanishing of all four boundary equations $\mathcal{B}_a$ guarantees the vanishing of the incoming constraints in the region of interest and is consistent with the propagation of the outgoing constraints along characteristics. In practice, this means that one can use $\mathcal{B}_a = 0$ to prescribe as many boundary values as possible, without worrying about inconsistencies with the initial values. This necessary analytic consistency should not be mistaken for numerical consistency, however. In fact, it is not known at this stage whether this analytic consistency is stable under small perturbations of either the initial data or the boundary data, a point that is critical to the numerical implementation. Additionally, how to proceed in order to investigate the stability of an initial-boundary-value problem with constraints remains, to our knowledge, an open problem not addressed in the standard reference literature of strongly hyperbolic systems such as [5].

In the case of formulations that do not guarantee stable constraint propagation, $\mathcal{B}_a = 0$ on the boundary still appeals to us as the next best thing. In fact, in such cases there are hardly any handles on boundary conditions. Notice that imposing $\mathcal{B}_a = 0$ on the boundary does not involve any manipulation of the choice of evolution equations; it does not affect the evolution equations themselves nor the system of propagation of the constraints. The converse is not true: any manipulation of the evolution equations (by adding linear combinations of the constraints in the usual manner) necessarily affects both the propagation of the constraints and the role of $\mathcal{B}_a = 0$ on the boundary. The interested reader is referred to [2] for a series of numerical studies of the effect of the manipulation of the evolution equations on constraint propagation exclusively, without imposing $\mathcal{B}_a = 0$.

Several issues remain open at this time. First and foremost, there remains the issue of whether the boundary equations $\mathcal{B}_a = 0$ are consistent with a well-posed initial-boundary-value problem in the analytic sense, that is: in the sense that the solutions at a later time are continuous functions of the initial data and the boundary free data. The answer to this question will vary among the different strongly hyperbolic formulations available. The answer may be relevant to numerical relativity because, even though well-posedness of the initial-boundary value problem does not guarantee a stable numerical implementation, it is considered as a necessary [5] or at least desirable feature of the continuous equations being implemented. Second, any issues that may arise exclusively in connection with the numerical implementation of $\mathcal{B}_a = 0$ –but that are otherwise irrelevant to the analytic initial-boundary value problem of the Einstein equations– are also open at this time and are worth pursuing, their details being strongly dependent on the particulars of the formulation at hand. From the numerical point of view, for instance, the presence of undifferentiated terms, –which is not relevant to our current argument and does not affect the well-posedness of first-order problems– is critical to numerical stability [5]. In this respect, the effect of the undifferentiated terms of the boundary equations $\mathcal{B}_a = 0$ on any issues of stability remains to be determined, being, almost certainly, critically dependent on

the particulars of the formulation at hand. Additional open issues at this time include checking for the consistency of $\mathcal{B}_a = 0$ with other boundary conditions already in use in numerical simulations, and, eventually, how the use of $\mathcal{B}_a = 0$ on the boundary may affect the run time of numerical simulations.

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